Mathematical Finance
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Assignment 8

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Barrier options

We consider in this exercise a frictionless financial market in continuous–time, without arbitrage opportunities, which contains one non–risky asset whose value at 0 is normalised to 1, and one risky asset with non–negative price process $(S_t)_{t\geq 0}$. The information flow is encoded in the filtration $\mathbb{F}$, which is simply generated by S. We fix a European style payoff function $f : \mathbb{R} \longrightarrow \mathbb{R}$, as well as some maturity $T > 0$, and a barrier $L > 0$. Barrier options are of 4 types

(i) An Up–and–In barrier option with payoff f and upper barrier L pays $f(S_T)$ at time T if the barrier L has been crossed from below before T. In other words, its payoff is $f(S_T)\mathbf{1}_{\{\sup_{0\leq t\leq T} S_t \geq L\}}$. Its value at time t will be denoted by $\mathrm{UI}_t(T, L; S, f(S_T))$.

(ii) An Up–and–Out barrier option with payoff f and upper barrier L pays $f(S_T)$ at time T if the barrier L has not been crossed from below before T. In other words, its payoff is $f(S_T)\mathbf{1}_{\{\sup_{0\leq t\leq T} S_t < L\}}$. Its value at time t will be denoted by $\mathrm{UO}_t(T, L; S, f(S_T))$.

(iii) A Down–and–In barrier option with payoff f and lower barrier L pays $f(S_T)$ at time T if the barrier L has been crossed from above before T. In other words, its payoff is $f(S_T)\mathbf{1}_{\{\inf_{0\leq t\leq T} S_t \leq L\}}$. Its value at time t will be denoted by $\mathrm{DI}_t(T, L; S, f(S_T))$.

(iv) A Down–and–Out barrier option with payoff f and lower barrier L pays $f(S_T)$ at time T if the barrier L has not been crossed from above before T. In other words, its payoff is $f(S_T)\mathbf{1}_{\{\inf_{0\leq t\leq T} S_t > L\}}$. Its value at time t will be denoted by $\mathrm{DO}_t(T, L; S, f(S_T))$.

We also denote by $P_t(T; S, f(S_T))$ the price at time $t \in [0, T]$ of the (standard) European option with maturity T and payoff $f(S_T)$. Finally, we say that a barrier option is regular if its payoff function is zero at and beyond the barrier. More precisely, for a Down type option this means that $f(x) = 0$ for $x \leq L$, and $f(x) = 0$ for $x \geq L$ for an In type option. When this is not the case, the barrier option is called reverse.

1) Show that for any $t \in [0, T]$

$$\mathrm{UI}_t(T, L; S, f(S_T)) + \mathrm{UO}_t(T, L; S, f(S_T)) = P_t(T; S, f(S_T)),$$
$$\mathrm{DI}_t(T, L; S, f(S_T)) + \mathrm{DO}_t(T, L; S, f(S_T)) = P_t(T; S, f(S_T)).$$

Deduce that as long as we know $P_t(T; S, f(S_T))$, it is sufficient to study barrier options of type In. We will therefore concentrate on these in the rest of the exercise.

2) Show that for any $t \in [0, T]$, we have

$$\mathrm{UI}_t(T, L; S, f(S_T)) = \mathrm{UI}_t\big(T, L; S, f(S_T)\mathbf{1}_{\{S_T<L\}}\big) + P_t\big(T; S, f(S_T)\mathbf{1}_{\{S_T\geq L\}}\big),$$
$$\mathrm{DI}_t(T, L; S, f(S_T)) = \mathrm{DI}_t\big(T, L; S, f(S_T)\mathbf{1}_{\{S_T>L\}}\big) + P_t\big(T; S, f(S_T)\mathbf{1}_{\{S_T\leq L\}}\big).$$

Deduce that as long as we know $P_t\big(T; S, f(S_T)\mathbf{1}_{\{S_T\geq L\}}\big)$ and $P_t\big(T; S, f(S_T)\mathbf{1}_{\{S_T\leq L\}}\big)$, it is sufficient to study regular barrier options of type In. We will therefore concentrate on these in the rest of the exercise, **meaning that the payoff f is from now on regular**.

3) From now on (that is until the end of the exercise), we consider the Black–Scholes model, that is to say that we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is directly assumed to be a risk–neutral measure, under which the dynamics of the unique risky asset S in the market is given by

$$S_t = S_0 + \int_0^t rS_s \mathrm{d}s + \int_0^t \sigma S_s \mathrm{d}B_s^{\mathbb{Q}},\ t \geq 0,$$

where $r \geq 0$ is the short–term interest rate. We define $\gamma := 1 - \frac{2r}{\sigma^2}$. Show that the process $(S_t^\gamma)_{t\geq 0}$ is an $(\mathbb{F}, \mathbb{Q})-$martingale (this is S to the power γ here). Deduce that we can define a probability measure $\widehat{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ such that the density, on $\mathcal{F}_t$, of $\widehat{\mathbb{Q}}$ with respect to $\mathbb{Q}$ is

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \frac{S_t^\gamma}{S_0^\gamma},\ t \in [0, T].$$

4) Using Girsanov's theorem, show that the following symmetry relationship holds

$$\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{S_t}\right)^\gamma f\left(\frac{S_t^2}{S_T}\right)\bigg|\mathcal{F}_t\right],\ t \in [0, T],$$

and deduce that

$$P_t(T; S, f(S_T)) = P_t\left(T; S, \left(\frac{S_T}{S_t}\right)^\gamma f\left(\frac{S_t^2}{S_T}\right)\right),\ t \in [0, T].$$

5) Fix now some $t \in [0, T]$. If $\sup_{0\leq s\leq t} S_s \geq L$, compute $\mathrm{UI}_t(T, L; S, f(S_T))$, and if $\inf_{0\leq s\leq t} S_s \leq L$, compute $\mathrm{DI}_t(T, L; S, f(S_T))$.

6) In this question we concentrate on the Up–and–In regular barrier option. Fix again some $t \in [0, T]$ and assume now that $\sup_{0\leq s\leq t} S_s < L$. We define then $\tau_L(t)$ to be the first instant after t when the risky asset price S goes above the barrier L. In other words

$$\tau_L(t) := \inf\big\{s \geq t : S_s \geq L\big\}.$$

Why do you have $S_{\tau_L(t)} = L$? Show then, using the tower property for conditional expectations[1], that

$$\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\mathbf{1}_{\{\sup_{0\leq s\leq T} S_s\geq L\}}\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[f(S_T)\mathbf{1}_{\{\tau_L(t)\leq T\}}\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{L}\right)^\gamma f\left(\frac{L^2}{S_T}\right)\bigg|\mathcal{F}_t\right].$$

Hint: do not forget to use the fact that the payoff function f is assumed to be regular, implying in the case of the Up–and–In option that $f(x) = 0$ for $x \geq L$.

7) In this question we concentrate on the Down–and–In regular barrier option. Fix again some $t \in [0, T]$ and assume now that $\inf_{0\leq s\leq t} S_s > L$. We define then $\rho_L(t)$ to be the first instant after t when the risky asset price S goes below the barrier L. In other words

$$\rho_L(t) := \inf\big\{s \geq t : S_s \leq L\big\}.$$

Why do you have $S_{\rho_L(t)} = L$? Show then, using the tower property for conditional expectations, that

$$\mathbb{E}^{\mathbb{Q}}\big[f(S_T)\mathbf{1}_{\{\inf_{0\leq s\leq T} S_s\leq L\}}f(S_T)\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\big[f(S_T)\mathbf{1}_{\{\rho_L(t)\leq T\}}f(S_T)\big|\mathcal{F}_t\big] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_T}{L}\right)^\gamma f\left(\frac{L^2}{S_T}\right)\bigg|\mathcal{F}_t\right].$$

8) Deduce from 6) and 7) that for any $t \in [0, T]$, when $\sup_{0\leq s\leq t} S_s < L$ we have

$$\mathrm{UI}_t(T, L; S, f(S_T)) = P_t\left(T; \left(\frac{S_T}{L}\right)^\gamma f\left(\frac{L^2}{S_T}\right)\right),$$

[1] Keep in mind that despite the fact that $\tau_L(t)$ is a random variable, you can use all the properties that you know, in particular you can condition with respect to $\mathcal{F}_{\tau_L(t)}$, and the event $\{\tau_L(t) \leq T\}$ belongs to $\mathcal{F}_{\tau_L(t)}$.

and when $\inf_{0\le s\le t} S_s > L$

$$\mathrm{DI}_t(T, L; S, f(S_T)) = P_t\left(T; \left(\frac{S_T}{L}\right)^{\gamma} f\left(\frac{L^2}{S_T}\right)\right).$$

Deduce a static replication strategy for the regular Up–and–In and Down–and–In barrier options (that is to say a replication strategy for these options which only invests in the market at the beginning of the period and then does not rebalance the portfolio until maturity). You can assume that all European options with maturity T are available to trade on the market.

9) Prove Proposition 7.6.10 in the lecture notes.